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Magnetohydrodynamic Mach Cones[†]

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Summary

Features of the surfaces of main disturbance created by a small object in steady motion through a conducting fluid are examined. These surfaces are found by drawing tangent cones from the object to the relevant wave-front diagrams. The outer wave cone (when present) is smooth, but the two inner cones have cross sections similar to the cusped figures of the inner wave-front diagram. It is conjectured that the disturbance may be concentrated along such line cusps. This has particular relevance in the application of known two-dimensional results to three-dimensional problems, say in the well-known techniques of aerodynamics. In MHD the omission of the large disturbance characteristics implicit in a two-dimensional solution may invalidate its use in any practical three-dimensional problem.

(1) Introduction

A BRIEF ACCOUNT is given here of the surfaces present in magnetohydrodynamic (MHD) flow analogous to the Mach cones of ordinary fluid dynamics. Numerical investigations of the shape of these surfaces have been made in several cases, giving an indication of the main features of the surfaces in the general case. Some tentative deductions from the character of these surfaces are made for the disturbance pattern of a small object in a steady MHD flow. It is thought unnecessary in a short paper to include a full introduction to a subject which is currently receiving considerable attention. The Proceedings of the Williamsburg Symposium provide substantial background material (in particular the papers by Sears[1] and Grad[2]).

The evolution of the disturbance pattern created by a point disturbance in an infinitely conducting fluid has been investigated from several viewpoints. Friedrichs and Kranzer[3] give the wave-front diagram, that is the surfaces of main disturbance. Friedlander[4] has obtained the orders of magnitude of the discontinuities across these wave-fronts for a class of initial disturbance. Weitzner[5] has derived the Green's function for the two-dimensional case. It is intended to use some of these results for time-dependent problems to predict the general features of the flow pattern of a small object in steady flow.

In the application of results of time-dependent flows to steady flows, a reliance on physical intuition is needed. Huygens' principle is used to obtain the surfaces of main disturbance as the envelope of the surfaces of successive small disturbances emitted from each position the body occupies. In steady motions the surfaces sent out at each instant are similar and the main disturbance surfaces are the tangent cones drawn to the wave-front diagram from a point representing the velocity of the motion (see, for example, Fig. 1). In ordinary fluid dynamics this construction yields the Mach cone, a right-circular cone to the spherical wave-front. Sears and Grad have indicated results of this approach for MHD. For instance, the rejection of the *nonphysical* nappe of each steady characteristic cone occasionally is consistent with forward-facing characteristics. However, this does not violate general ideas of the nature of the disturbance field. Extensive work has been done on two-dimensional, linearised, MHD, steady-flow problems (see, for example Grad,[2] McCune and Resler,[6] Cumberbatch, Sarason, and Weitzner[7]) with implicit use of the above ideas.

This paper examines the characteristic cone surfaces arising in three-dimensional steady flow. Section (2) contains details of a procedure whereby the shape of the cones may be obtained, with emphasis on a numerical approach. A description of the general features of the surfaces obtained for specific examples is given, together with figures of cross sections of the cones. Use of Friedlander's theory to predict the order of magnitude of the discontinuities across the characteristic surfaces is made in Section (3). It is conjectured there that these results have particular significance for future work since they imply severe restrictions on the use of two-dimensional theory, and full three-dimensional investigations are made necessary
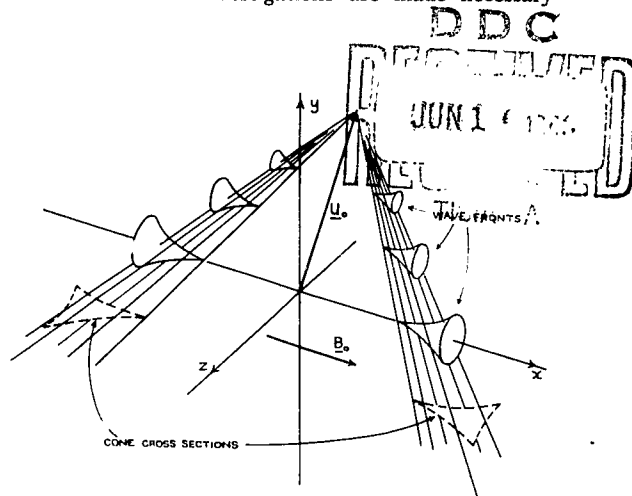

FIG. 1. Characteristic cones in steady MHD flow obtained as envelopes of wave-fronts emitted at each instant of time. Only the tangent cones to the inner wave-fronts are illustrated. A cross-section in the $x - v$ plane s shown more accurately in Fig. 2.
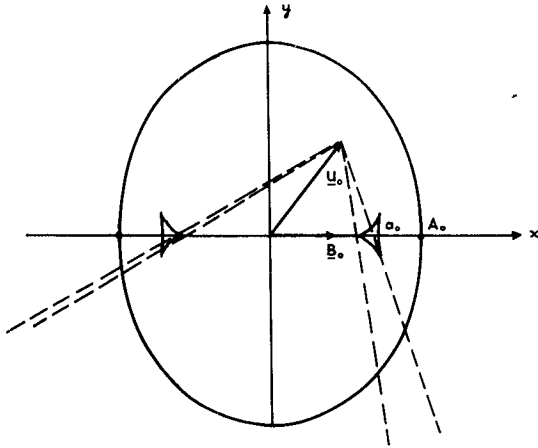
FIG. 2. Intersections with the $y - x$ plane of the wave-fronts (solid lines) and the tangent cones from U_0 (dashed lines). The wave-fronts represent the case $A_0 = \sqrt{2}\, a_0$.

for practical flow problems. A possible application of MHD steady flow is the motion of a satellite through the ionosphere and this flow has been discussed by Lighthill.[8] Interest there was confined to the case of high magnetic pressure and a limiting case was obtained in which the characteristic cone system reduces to two lines trailing from the body in a V-pattern.

(2) Tangent Cones

A system of axes is chosen with the x-axis along the vector B_0 representing the undisturbed magnetic field, and the x-y plane as the plane defined by B_0 and U_0, the velocity of the object. The wave-front surfaces resulting from an isolated disturbance at the origin consist of a convex outer surface similar to the spherical acoustic surfaces but elongated perpendicular to the magnetic field, and two inner cusped conoids propagating each side of the origin in narrow cones with axes along the magnetic field. These surfaces have B_0 as an axis of revolution and their intersection with the x-y plane is shown in Fig. 2. The Alfvén waves excited travel along the B_0 lines and are represented by the points $\pm A_0$ on the x-axis, where

$$A_0 = |B_0|/\sqrt{\mu_0\,\rho_0} \qquad (1)$$

μ_0 and ρ_0 being the magnetic permeability and density of the fluid. These waves coincide with the waves of the inner and outer fronts with normals along the x-axis, being located on the outer front for $A_0 > a_0$ and on the inner front for $A_0 < a_0$. The Alfvén velocity A_0 determines the scale of Fig. 2 and allows the velocity U_0 to be introduced there.

The surfaces of main disturbance in steady motion of a point object are assumed to be obtained by constructing tangent cones from U_0 to the wave-front surfaces. This assumption is based upon physical ideas and the rigorous mathematical verification re-

mains to be done. In this respect it is noted that the present work meets difficulties in deciding whether the line joining the inner conoids is part of the wave-front. However in the fully hyperbolic flow regions it may be shown that this construction is correct and that the line joining the inner conoids is part of the characteristic locus.* The two tangent cones to the inner wave-fronts would then be connected by a sheet surface. It is apparent that two cones to the inner fronts always exist (see Fig. 2 where the intersections of these cones with the x-y plane are shown) but a further cone (forward facing) is evident if U_0 lies inside either of the inner fronts. This cone disappears as U_0 moves outside this region and reappears as U_0 moves outside the outer wave-front. The disappearance of this cone in the region between the inner and outer wave-fronts corresponds to a change in character of the governing differential equations from hyperbolic to a mixture of hyperbolic and elliptic type. The Alfvén part of the disturbance lies on the lines from U_0 through the points $\pm A_0$ on the x-axis.

The wave speed c and the coordinates x, $r = \sqrt{y^2 + z^2}$ of the wave-fronts may be found from the following parametric relations (see Ref. 3 or Ref. 9), expressed in a form suitable for computation. Writing

$$c = A_0 C, \qquad x = A_0 X, \qquad r = A_0 R$$

and

$$f^2 = 1 + \alpha^2 - 2\alpha \cos 2\phi$$

where

$$\alpha = a_0^2/A_0^2$$

then

$$\left.\begin{array}{l} C^2 = (1/2)(1 + \alpha \mp f) \\[4pt] X = (C \pm R \sin \phi)\sec\phi \\[4pt] R = C^3 f^{-1} \sin \phi \end{array}\right\} \qquad (2)$$

In the above, a_0 is the sound speed and the upper signs relate to the inner fronts and the lower signs to the outer front. The wave-fronts are obtained in $x \geqslant 0$

SECTION OF INNER WAVE FRONT

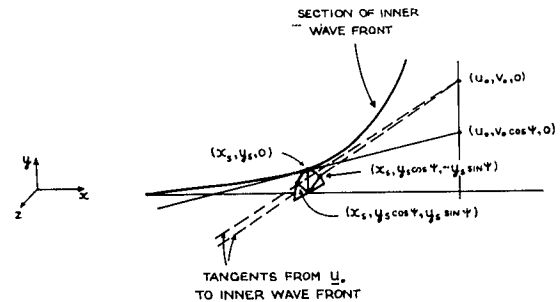TANGENTS FROM U_0 TO INNER WAVE FRONT

FIG. 3. Tangents to the inner wave-front from $U_0 = (u_0, v_0, 0)$ obtained by first drawing the tangent in the $x - y$ plane from $(u_0, v_0 \cos\psi, 0)$.
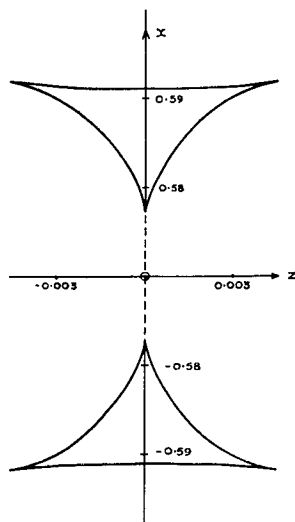
FIG. 4. Cross-section of the cones from a velocity vector $U_0 = A_0(0, 0.3, 0)$ to the inner wave-fronts for the case $A_0 = \sqrt{2}\, a_0$. The cross-section is taken in the plane $y = 0$—i.e. perpendicular to the velocity vector. To allow a view of the shape of the cones, a collapsed scale in the x-direction between the cones is used and is indicated by the dashed line.

for $0 \leqslant \phi \leqslant \pi/2$ and in $x < 0$ by reflection. The parametric angle ϕ is the angle between B_0 and the normal to the wave-front traveling with velocity c. Hence the intersections of the wave-fronts with the first quadrant of the x-y plane are curves with tangents given by

$$dy/dx = \pm \cot \phi \qquad (3)$$

The tangent cones from a point $U_0 = (u_0, v_0, 0)$ to the surfaces defined by Eqs. (2) are now obtained. This is done by reducing the construction to a two-dimensional one in the x-y plane. The normal to the tangent has components $(-\sin \phi, \cos \phi, 0)$ and for tangency at a point $(x_0, y_0, 0)$ on the wave-front surface

$$(x_0 - u_0) \sin \phi - (y_0 - v_0 \cos \psi)\cos \phi = 0$$

is satisfied. This condition can be rewritten to show that the tangent from $(u_0, v_0, 0)$ to the wave-front is tangent at $(x_0, y_0 \cos \psi, y_0 \sin \psi)$ where the normal has components $(-\sin \phi, \cos \phi \cos \psi, \cos \phi \sin \psi)$. The tangent cone is obtained by variation of ψ. This construction is illustrated in Fig. 3 with reference to the inner front and can be seen to be the determination of the intersection of the curve of tangency on the wave-front surface with a plane through the x-axis at an angle ψ to the y-axis. A convenient numerical procedure is the reverse operation. That is, for a point $(x_0, y_0, 0)$ on the wave-front, the tangent line through it is found using Eq. (3), and its intersection with the ordinate line to the point $(u_0, v_0, 0)$ then determines the angle ψ. In particular, for the tangent cone to the inner front in the first quadrant, the equation

$$v_0 \cos \psi = y_s + \cot \phi (u_0 - x_s) \qquad (4)$$

is solved for ψ for a succession of points $(x_s, y_s, 0)$ on the slow wave, resulting in the lines

$$\frac{x - u_0}{u_0 - x_s} = \frac{y - v_0}{v_0 - y_s \cos \psi} = \frac{-z}{y_s \sin \psi} \qquad (5)$$

on the tangent cone. This procedure is easily extended to cover the wave-fronts in the whole x-y plane and it is suitable mainly for programming on an electronic computer. [Eq. (4) has a solution for ψ only over parts of the wave-fronts and the subsidiary calculations are redundant elsewhere.]

The numerical procedure described above was performed for several cases; the results of two calculations are shown in Figs. 4 and 5. Similar characteristics were found in all the cases and their general features will now be described. The cone to the outer front (when present) has a smooth oval shape in cross section. From points inside the inner front the forward-facing cone is also smooth. It is noted that this cone may be tangent at the inner front conoid partly on the "base" (concave to $x = \infty$) and partly on the convex surface. The main interest lies in the cones to the inner fronts which can be drawn from any point. These cones were found to have cross sections similar to the inner wave-front shapes of Fig. 2 (see Fig. 4). The inward-facing cusps of these cross sections arise from the tangents near the cusps of the inner fronts on the x-axis. It is assumed here that the tangents near these cusps have the line through the cusp as a limit which closes the cone in a line cusp (this point will be discussed later). The lines through the wave-front cusps have been drawn in Fig. 2 together with the two-dimensional tangents to the wave-fronts (which form the outer tangents in the plane of symmetry of the cones). The cone cross sections were found to be convex to the cone interior and hence two further cusps are present on each cone. These cusps are unrelated to any of the cusps on the wave-front surfaces.

(3) Discussion

Friedlander's results estimating the order of magnitude of the disturbance on a wave-front may be used to predict the disturbance pattern on the cones obtained above. This work assumed a nonexcitation of Alfvén
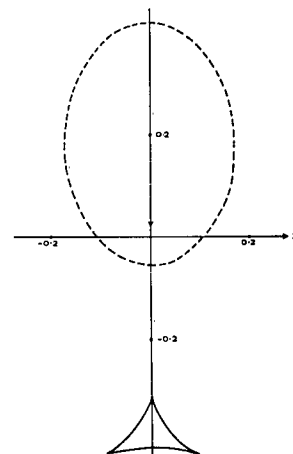

FIG. 5. Cross-section of the cones from $U_0 = A_0$ (0.69, 0.06, 0) to the inner wave-fronts for $A_0 = \sqrt{2}\, a_0$. The cross-section is taken in a plane through the origin perpendicular to U_0. Here U_0 lies inside the inner front and the forward-facing cone has been traced back to give an intersection, which is shown (1/20 size) by the dashed curve. Note that one of the cones is very small, being represented by the arrowhead above the origin.

waves, but the general solution is assumed to have the main features of this restricted solution. A disturbance initially confined to $|\mathbf{r}| < \epsilon$ was found to propagate as a front of order $\epsilon |\mathbf{r}|^{-1}$ and width ϵ on the smooth sections of the inner and outer wave-fronts. However, the disturbances near the cusps of the inner front on the x-axis are of order $\epsilon^{7/3}|x|^{-1/3}$ and are spread over a region of order $\epsilon^{2/3}|x|^{1/3}$. The residual disturbance left after the passage of the fronts is of order $\epsilon^3|\mathbf{r}|^{-3}$. Hence, at large distances, the disturbance is concentrated in the neighborhoods of the point cusps on the inner wave-fronts. The associated line cusps of the cones to the inner fronts therefore carry the dominant part of the disturbance created by a body in steady motion. The main disturbance will lie along a V-pattern trailing from the body, obtained by drawing lines from $\mathbf{U}_0$ through the points $\pm a_0 A_0 (a_0^2 + A_0^2)^{-1/2}$ on the x-axis. This pattern coincides with that obtained by Lighthill in the large-magnetic-field limit where the whole characteristics system has degenerated to these two lines.

It is evident that a formal two-dimensional solution, which assumes characteristics drawn tangent to the two-dimensional wave-fronts, omits the tangents lying in the plane of interest drawn through the inner cusps. That is, referring to Fig. 2 as a two-dimensional characteristics diagram, the lines drawn through the inner cusps there as part of the three-dimensional system in the x-y plane are not characteristics derivable from a two-dimensional theory. The previous two-dimensional work has not included these characteristics (see, for example, the characteristics diagrams drawn in Refs. 1, 2, 6, and 7). Moreover, since these particular characteristics are the main disturbance carriers, the use of two-dimensional theory (say in the flow over wings) will be suspect since any three-dimensional influences will be dominant. It is conjectured therefore that the practical value of the present two-dimensional solutions is highly doubtful, and an analytical study of three-dimensional steady flow will be required for any applications.

The discussion so far has been confined to nondissipative small disturbance theory, which has ascertainable limitations in scope and application. The effect of nonlinear and dissipative forces to sharpen or smear the disturbances calculated—e.g., by Friedlander—is of interest. For instance, the dissipative effect of a small nonzero value of the electrical resistance is to diffuse the wave patterns lying along the characteristic surfaces (see—e.g., Sears). Of particular interest is the effect of these forces on the dominant part of the disturbance centered along the line cusps of the inner cones. The nature of the disturbance across the line joining the cusps of the inner conoids is also important since it is relevant to conjecture whether the nonlinear and dissipative forces would confine the disturbance pattern to the neighborhoods of the cones obtained (giving two separated inner wave disturbances dominated by the V-pattern along the line cusps), or to another surface—e.g., a surface bounding both cones and the sheet joining the line cusps. This latter case would have a dumbbell shape in cross section.

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